The Evolving Influence of Psychometrics in Political Science

by

Keith T. Poole
University of California, San Diego

Psychometrics is a subfield of Psychology devoted to the development, evaluation, and application of mental tests of various kinds. These mental tests attempt to measure knowledge, attitudes, personality traits, and abilities. Psychometrics has its origins in the work of Sir Francis Galton (1822 – 1911), Karl Pearson (1857 – 1936), and Charles Spearman (1863 – 1945) in the late 19th and early 20th centuries. Galton's most famous work was *Hereditary Genius* (1869) in which he studied "illustrious" intellects and their families. His biographical data of the descendants of these illustrious intellects showed "regression to the mean" for a number of mental and physical characteristics that he regarded as important. Much of his work in the latter part of the 19th Century was devoted to eugenics. Galton was interested in measurement and developed a measure of *co-relation* which influenced the development of the correlation coefficient by Karl Pearson. He and Karl Pearson founded the journal *Biometrika* in 1901.

Galton was a major influence on both Karl Pearson and Charles Spearman. Pearson began his professional life as an attorney from 1881 to 1884 but in 1884 he was appointed as a professor of applied mathematics and mechanics at University College, London. He became professor of eugenics in 1911 and was the editor of *Biometrika* from 1902 to 1936. Pearson invented the product moment correlation coefficient which is universally denoted as *r* and he should also be credited with the invention of Principal

Components Analysis (what we now would think of as straightforward eigenvalue/eigenvector decomposition). Pearson called it "the method of principal axes" and states the problem quite succinctly: "In many physical, statistical, and biological investigations it is desirable to represent a system of points in plane, three, or higher dimensioned space by the 'best-fitting' straight line or plane" (1901, p. 559). Remarkably, this also describes the essence of the famous Eckart-Young theorem (Eckart and Young, 1936, see below) which is the foundation of general least squares problems (Lawson and Hanson, 1974).

Charles Spearman came late to the study of psychology. He began his professional career as an officer in the British army and he served in the 1885 Burmese war and in the Boer War in South Africa. He was 43 years old in when he earned his Ph.D. in psychology at Leipzig in 1906. He held chaired professorships at University College London from 1907 to 1931.

While still a graduate student he published his famous 1904 paper that used *factor analysis* to analyze a correlation matrix between test scores of twenty-two English high school boys for Classics, French, English, Math, Pitch, and Music. This correlation matrix is shown in Table 1.

Table 1: Spearman's 1904 (Rank Order) Correlation Matrix

Classics	**1.00**					
French	**.83**	**1.00**				
English	**.78**	**.67**	**1.00**			
Math	**.70**	**.67**	**.64**	**1.00**		
Pitch	**.66**	**.65**	**.54**	**.45**	**1.00**	
Music	**.63**	**.57**	**.51**	**.51**	**.40**	**1.00**

This correlation matrix is historic for two reasons. First, Spearman computed a form of rank-order correlation between each pair of skills across the twenty-two school boys.[1] Second, he applied factor analysis to the matrix of correlations to extract a common or *general* (the “g” factor) factor from the matrix. His method of extracting the g factor was based on his method of “tetrad differences”. A tetrad difference is actually the determinant of a 2 by 2 matrix and if there is only one factor then these differences should all be close to zero. For example, using English and Math, the tetrad difference is .78*.67 - .67*.70 or .054. If the tetrad differences are all close to zero then the matrix only has one factor (rank of one). Spearman derived an elaborate formula for extracting this g factor from a correlation matrix.[2] The notorious Cyril Burt tried to claim that he invented factor analysis after Spearman’s death. However, there is no question that Spearman was the inventor (Lovie and Lovie, 1993).

Lewis Leon Thurstone (1887 – 1955) thought Spearman’s one factor theory of intelligence was wrong. Thurstone was a polymath who earned an engineering degree at Cornell in 1912 and a PhD in Psychology at Chicago in 1917 where he became a professor from 1924 to 1952. While an engineering student he invented a flicker-free motion picture projector and briefly worked as an assistant to Thomas Edison in 1912. He made many fundamental contributions to psychological science the most important of which were multiple factor analysis and the *law of comparative judgment*.

Thurstone generalized Spearman’s tetrad differences approach to examine higher order determinants and succeeded in developing a method for extracting multiple factors from a correlation matrix (Thurston, 1931; 1947). Thurstone’s theory of intelligence postulated seven rather than one primary mental ability and he constructed tests specific to the seven abilities: verbal comprehension, word fluency, number facility, spatial visualization, associative memory, perceptual speed, and reasoning (Thurstone, 1935).

Thurstone also developed the law of comparative judgment. Thurstone's law is more accurately described as a measurement model for a *unidmensional subjective continuum*. Subjects are asked to make a series of n(n-1)/2 pairwise comparisons of n stimuli. It is assumed that a subject's response reflects the momentary subjective value associated with the stimulus, and that the probability distribution of these momentary values is normally distributed. It is then possible to recover the underlying continuum or scale by essentially averaging across a group of subjects. If the variances of the stimuli (the *discriminal* dispersions) on the underlying scale are the same (Case 5 of the model), this is equivalent to the requirement of parallel item characteristic curves in the Rasch model. Case 5 of Thurstone's method should yield essentially the same results as the Rasch model for dichotomous data (Andrich, 1978).

Although Thurstone developed multiple factor analysis it was Harold Hotelling (1895 – 1973) who gave principal components a solid statistical foundation (Hotelling, 1933). Hotelling had an eclectic background. He received a BA in journalism in 1919 from the University of Washington and a Ph.D. in mathematics from Princeton in 1924. Reflecting this eclectic background, Hotelling made fundamental contributions in both economics and statistics. In economics Hotelling's famous 1929 paper on the stability of competition is generally recognized as the beginnings of the spatial (geometric) model of voting (see below). It introduced the simple but profound idea that if there are two stores on a street then it is in the interest of each store to locate in the middle (the median walking distance) of the street where each gets one half of the market. Two years later in a 1931 paper Hotelling laid out what has since become labeled "confidence intervals" in an analysis of the use of the Student's t distribution for hypothesis testing.

Hotelling was one of a number of distinguished mathematicians and physicists who made fundamental contributions to the development of psychometrics in the 1930s and 1940s. As fate would have it a number of these contributors were at the University of Chicago at the same time as Thurstone. Thurstone was the main force behind the founding of the Psychometric Society and its journal *Psychometrika* (Takane, 2004). Carl H. Eckart (1902 – 1973), a distinguished quantum physicist (Munk and Preisendorfer, 1976), and Gale Young, an applied mathematician, published their landmark paper "The Approximation of One Matrix by Another of Lower Rank" in the very first issue of *Psychometrika* in 1936. Formally, the Eckart-Young theorem is:

Given a n by m matrix **A** of rank $r \leq m \leq n$, and its singular value decomposition, $\mathbf{U\Lambda V'}$, where **U** an n by m matrix, **V** is an m by m matrix such that $\mathbf{U'U=V'V=VV'=I}$, and $\mathbf{\Lambda}$ is an m by m diagonal matrix with the singular values arranged in decreasing sequence on the diagonal

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \lambda_m \geq 0$$

then there exists an n by m matrix **B** of rank s, $s \leq r$, which minimizes the sum of the squared error between the elements of **A** and the corresponding elements of **B** when

$$\mathbf{B} = \mathbf{U\Lambda_s V'}$$

where the diagonal elements of $\mathbf{\Lambda_s}$ are

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \lambda_s > \lambda_{s+1} = \lambda_{s+2} = \ldots = \lambda_m = 0$$

The Eckart-Young theorem states that the least squares approximation in s dimensions of a matrix **A** can be found by replacing the smallest m-s roots of $\mathbf{\Lambda}$ with zeroes and remultiplying $\mathbf{U\Lambda V'}$. This Theorem was never *explicitly* stated by Eckart and

Young. Rather, they use two theorems from linear algebra (the key theorem being singular value decomposition[3]) and a very clever argument to show the truth of their result. Later, Keller (1962) independently rediscovered the Eckart-Young theorem.

The Eckart-Young theorem provides a formal justification for the selection of the number of factors in a factor analysis (as well as many other general least squares problems). The Eckart-Young theorem along with the results of Gale Young and Alston Householder (1904 - 1993) published in *Psychometrika* in 1938 provided the foundations for *classical multidimensional scaling*.

Multidimensional scaling (MDS) methods represent measurements of similarity between pairs of stimuli as distances between points in a low-dimensional (usually Euclidean) space. The methods locate the points in such a way that points corresponding to very similar stimuli are located close together while those corresponding to very dissimilar stimuli are located further apart. Warren Torgerson (1924 –1997) in a 1952 *Psychometrika* paper showed a simple method of MDS based upon the work of Eckart and Young (1936) and Young and Householder (1938) (see also Torgerson, 1958). The method is elegantly simple. First, transform the observed similarities/dissimilarities into squared distances. (For example, if the matrix is a Pearson correlation matrix subtract all the entries from 1 and square the result.) Next, *double-center* the matrix of squared distances by subtracting from each entry in the matrix the mean of the row, the mean of the column, adding the mean of the matrix, and then dividing by -2. This has the effect of removing the squared terms from the matrix leaving just the cross-product matrix (see Gower, 1966). Finally, perform an eigenvalue-eigenvector decomposition to solve for the coordinates.

For example, suppose there are n stimuli and let **D** be the n by n symmetric matrix of squared distances between every pair of the stimuli. Let **Z** be the n by s matrix of coordinates of n points in an s-dimensional Euclidean space that represent the n stimuli and let **Y** be the n by n double centered matrix. The elements of **Y** are:

$$y_{ij} = \frac{(d_{ij}^2 - d_{.j}^2 - d_{i.}^2 + d_{..}^2)}{-2} = (z_i - \bar{z})'(z_j - \bar{z})$$

where $d_{.j}^2$ is the mean of the jth column, $d_{i.}^2$ is the mean of the ith row, $d_{..}^2$ is the mean of the matrix, z_i and z_j are the s length vectors of coordinates for the ith and jth stimuli, and $\bar{z}$ is the s length vector of means for the n stimuli on the s dimensions. Without loss of generality the means can be set equal to zero so that the double-centered matrix is simply

$$\mathbf{Y} = \mathbf{ZZ'}$$

Let the eigenvalue-eigenvector decomposition be $\mathbf{U\Lambda U'}$; hence the solution is:

$$\mathbf{Z} = \mathbf{U\Lambda^{1/2}}$$

Torgerson's method is very elegant but similarities/dissimilarities data are rarely measured on a ratio scale. Indeed, it is very likely that data gathered from subjects is at best on an ordinal scale. Roger Shepard (1958) argued that the relationship between the true distance between a pair of stimuli and the observed distance was exponential. That is, if d is the distance between two stimuli then the reported similarity, δ, tends to be e^{-kd}, where k (k > 0) is a scaling constant (Shepard, 1958; 1963, 1987; Gluck, 1991; Nofosky, 1992; Cheng, 2000). This is known as a *response function*. Within Psychology, surveys and experiments of how people make similarities and preferential choice judgments show that very simple geometric models appear to structure responses to these tasks (Shepard, 1987). When individuals make a judgment of how similar two stimuli are, they appear to base the judgment upon how close the two stimuli are in an abstract psychological space (Nosofsky, 1984; 1992; Shepard, 1987; Gluck, 1991). The dimensions of these psychological spaces correspond to the attributes of the stimuli. A strong regularity is that these psychological spaces are *low dimensional* – very rarely above two dimensions – and that either the stimuli judgments are additive – that is, a city-block metric is being used – or simple Euclidean (Garner, 1974; Shepard 1987; 1991; Nosofsky, 1992).

Shepard's belief that response functions were exponential led him to develop *nonmetric* multidimensional scaling in which distances are estimated that reproduce a weak monotone transformation (or *rank ordering*) of the observed dissimilarities (Shepard, 1962a,b). Graphing the "true" (that is, the estimated or reproduced) distances – the d's – versus the observed dissimilarities – the δ's – revealed the relationship between them. This became known as the "Shepard diagram."

Shepard's program worked but the key breakthrough was Joseph Kruskal's idea of *monotone regression* that lead to the development of a powerful and practical nonmetric MDS program (Kruskal, 1964a,b; 1965). By the early 1970s this was known under the acronym KYST (Kruskal, Young, and Seery, 1973) and is still in use today.

MDS methods can be seen as evolving from factor analysis and Thurstone's unidimensional scaling method with the key difference being that MDS methods are applied to *relational* data, that is, data such as similarities and preferential choice data that can be regarded as *distances*. At the same time that MDS methods were evolving Louis Guttman (1916 – 1987) during the Second World War developed *scalogram analysis* or what is more commonly known as *Guttman Scaling* (Guttman, 1944, 1950).

A Guttman scale is the basis of all modern skills based tests. It is a set of items (questions, problems, etc.) that are ranked in order of difficulty so that those who answer correctly (agree) on a more difficult (or extreme) item will also answer correctly (agree) will all less difficult (extreme) items that preceded it.[4] Rasch analysis (more broadly, *item response theory*) is essentially a sophisticated form of Guttman scalogram analysis. They are techniques for examining whether a set of items is *consistent* in the sense that they all measure increasing/decreasing levels of some unidimensional attribute (e.g., mathematical ability; racial prejudice, etc).

At the same time that Torgerson was developing classical scaling and Guttman was developing scalogram analysis, Clyde Coombs (1912 – 1988) developed *unfolding analysis* (Coombs, 1950; 1952; 1958; 1964). Coombs was a student of Thurstone's and received his Ph.D. from the University of Chicago in 1940. After World War II Coombs became interested in preferential choice problems where the data consists of subjects' rank orderings of stimuli (Tversky, 1992). Coombs came up with the idea of an *ideal point* and a *single-peaked preference function* to account for the observed rank orderings. The idea was to arrange the individuals' ideal points and points representing the stimuli along a scale so that the distances between the ideal points and the stimuli points reproduced the observed rank orderings. Coombs called this an unfolding analysis because the researcher must take the rank orderings and "unfold" them. An individual's rank ordering is computed from her ideal point so that the reported ordering is akin to picking up the dimension (as if it were a piece of string) at the ideal point so that both sides of the dimension fold together to form a line with the individual's ideal point at the end.

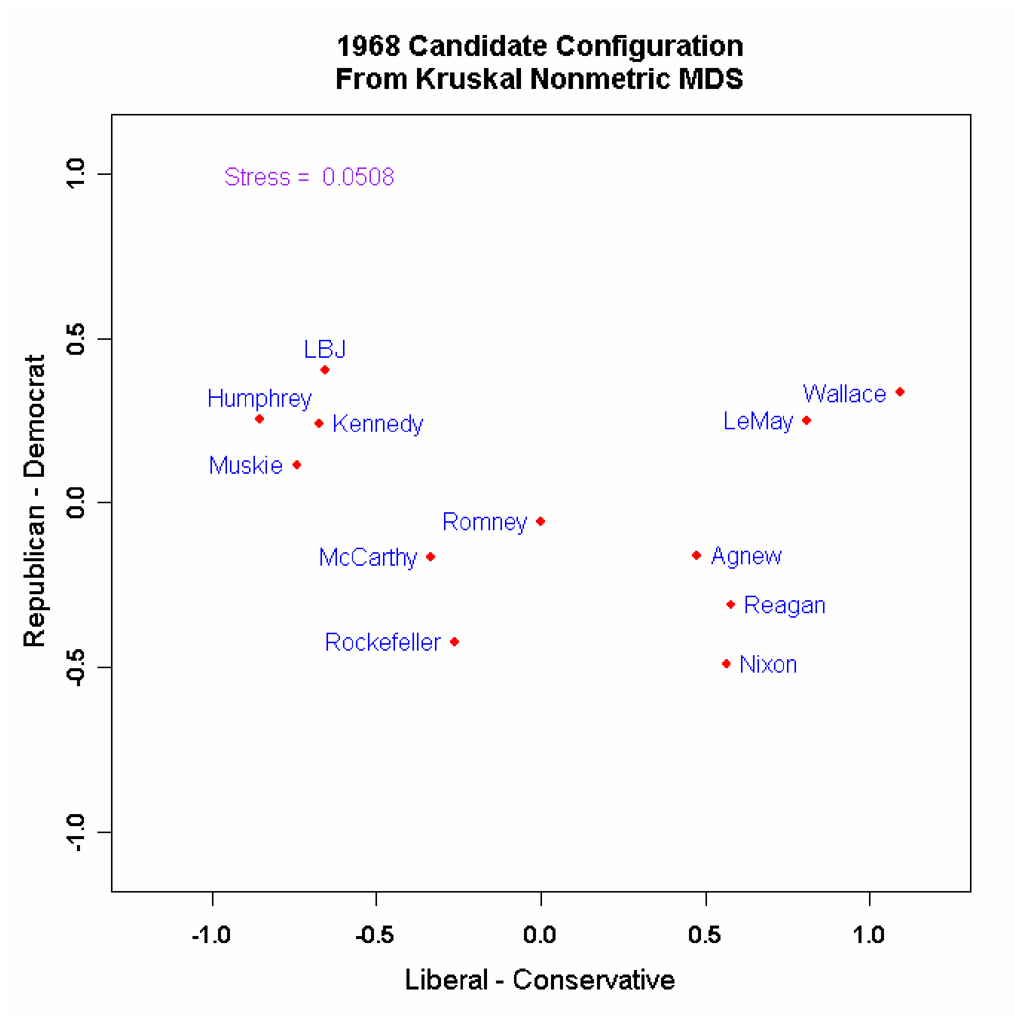
Unfolding analysis deals with relational data and is therefore an MDS method. Both unfolding analysis and scalogram analysis deal with individual's responses to a set of stimuli. But Guttman's model is very different from the unfolding model. In terms of utility theory unfolding analysis assumes a single-peaked (usually symmetric) utility function. That is, utility (the degree of preference) declines with distance from the individual's ideal point. In contrast, Guttman scaling is based on a utility function that is always monotonically increasing or decreasing over the relevant dimension or space. Above some threshold the individual always responds Yes/correct, and below the threshold the individual always responds No/incorrect. The counterpart to an ideal point is the position on the scale where the individual's responses switch from Yes/correct to No/incorrect.

Interestingly, these two very different models are observationally equivalent in the context of Parliamentary voting (Weisberg, 1968; Poole, 2005). In the unfolding model there are two outcomes for every Parliamentary motion – one corresponding to Yea and one corresponding to Nay. Legislators vote for the option closest to their ideal points. In one dimension this forms a perfect scalogram (Weisberg, 1968). Hence, Guttman scaling methods and their item response theory (IRT) parametric descendants can be used to analyze Parliamentary (binary choice) data.

By the mid 1950s factor analysis, Guttman scalogram analysis, and Thurstone's scaling methods had been developed and began to influence political scientists. Duncan MacRae's pathbreaking work on congressional voting (MacRae, 1958; 1970) utilized both factor analysis and unidimensional scaling methods at a time when computing resources were very primitive. MacRae used factor analysis to analyze correlation matrices computed between roll calls (usually Yule's Q's) and correlation matrices between legislators to uncover the dimensional structure of roll call voting. By analyzing the Yule's Q results MacRae was able to construct unidimensional scales for specific issue areas. MacRae proposed the model of roll call voting that Howard Rosenthal and I implemented as NOMINATE -- namely, ideal points for legislators and two policy outcomes per roll call, one for Yea and one for Nay. There is no doubt that MacRae would have estimated this model if computing resources in the 1950s had been up to the task.

Herbert Weisberg in his 1968 Ph.D. dissertation (Weisberg, 1968) systematically detailed the interrelationships of existing multivariate methods (most of which came from psychology) that had been used to analyze roll call voting. In his analysis of factor analysis, Guttman scaling, similarities analysis, and cluster analysis Weisberg showed the observational equivalence of the ideal point *proximity* model with the Guttman scalogram *dominance* model, and outlined a general framework for analyzing roll call voting.

In the late 1960s and early 1970s nonmetric MDS began to be used in political science. Beginning in 1968 feeling thermometers were included in the National Election Studies conducted by the University of Michigan. A feeling thermometer measures how warm or cold a person feels toward the stimulus; the measure ranges from 0 – very cold and unfavorable opinion – to 100 – very warm and favorable opinion -- with 50 as a neutral point. In 1968 respondents were asked to give feeling thermometer ratings to the presidential candidates George Wallace, Hubert Humphrey, and Richard Nixon, along with their vice presidential running mates and six other political figures. Herbert Weisberg and Jerrold Rusk (1970) computed Pearson correlations between every pair of political figures across the respondents and then used Kruskal's nonmetric MDS procedure (Kruskal, 1964a,b) to recover a candidate configuration. This configuration is shown in the figure below.



The availability of the feeling thermometer data led to efforts to apply unfolding methods to them directly. In these models the thermometers were regarded as inverse distances. For example, by subtracting them from 100 these transformed scores could be treated as distances between points representing the candidates and points representing the respondents. Techniques to perform unfolding analyses were developed by psycholometricians in the 1960s (Chang and Carroll, 1969; Kruskal, Young, and Seery,

1973) but the first application of unfolding to thermometers was done by George Rabinowitz (1973; 1976) using his innovative line-of-sight method. Almost at the same time Cahoon (1975) and Cahoon, Hinich, and Ordeshook (1976; 1978), using a statistical model based directly on the spatial model of voting (Davis and Hinich, 1966; 1967; Davis, Hinich, and Ordeshook, 1970; Enelow and Hinich, 1984), also analyzed the 1968 feeling thermometers. Later Poole and Rosenthal (1984), and Brady (1990) developed unfolding procedures that they applied to thermometer scores. Poole (1981; 1984; 1990) and Poole and Daniels (1985) also applied an unfolding procedure to interest group ratings of members of Congress.

In the 1980s political scientists began combining techniques from econometrics and statistics with approaches developed by psychometricians. Henry Brady made contributions to the statistical foundations of nonmetric MDS (Brady, 1985a) as well as methods for and problems with the analysis of preferences (Brady, 1985b; 1989; 1990). Poole and Rosenthal combined the random utility model developed by economists (McFadden, 1976), the spatial model of voting, and alternating estimation methods developed in psychometrics (Chang and Carroll, 1969; Carroll and Chang, 1970; Young, de Leeuw, and Takane, 1976; Takane, Young, and de Leeuw, 1977)[5] to develop NOMINATE, an unfolding method for parliamentary roll call data (Poole and Rosenthal, 1985; 1991; 1997; Poole, 2005).

The NOMINATE model is based on the spatial theory of voting. Legislators have ideal points in an abstract policy space and vote for the policy alternative closest to their ideal point. Each roll call vote has two policy points – one corresponding to Yea and one to Nay. Consistent with the random utility model, each legislator's utility function consists of (1) a *deterministic* component that is a function of the distance between the legislator and a roll call outcome; and (2) a *stochastic* component that represents the idiosyncratic component of utility. The deterministic portion of the utility function is assumed to have a normal distribution and voting is probabilistic. An alternating method is used to estimate the parameters. Given starting estimates of the legislator ideal points the roll call parameters are estimated. Given these roll call parameters, new legislator ideal points are estimated, and so on. Classical methods of optimization are used to estimate the parameters.[6]

In the 1990s and the early 2000s the availability of cheap, fast computers made simulation methods for the estimation of complex multivariate models practical for the first time[7] and these methods were fused with long standing psychometric methods. Specifically, *Markov Chain Monte Carlo* (MCMC) simulation (Metropolis and Ulam, 1949; Hastings, 1970; Geman and Geman, 1984; Gelfand and Smith, 1990; Gelman, 1992) within a Bayesian framework (Gelman, Carlin, Stern, and Rubin, 2000; Gill, 2002) can be used to perform an unfolding analysis of parliamentary roll call data. The general Bayesian MCMC method was introduced into Political Science by Andrew Martin and Kevin Quinn (Schofield, Martin, Quinn, and Whitford, 1998; Quinn, Martin, and Whitford, 1999; Martin and Quinn, 2002; Quinn and Martin, 2002; Martin, 2003; Quinn, 2004) and Simon Jackman (2000a; 2000b; 2001; Clinton, Jackman, and Rivers, 2004).

The primary application of Bayesian MCMC methods in political science has been to unfolding roll call data from legislatures and courts. Like NOMINATE, the foundation is the spatial theory of voting and the random utility model. This unfolding approach also uses an alternating structure only it consists of sampling from conditional distributions for the legislator and roll call parameters. Technically, this is *alternating conditional sampling* or the *Gibbs sampler* (Geman and Geman, 1984; Gelfand and Smith, 1990). Thus far the Bayesian MCMC applications have used a quadratic deterministic utility function with most of the applications being one dimensional. With a quadratic deterministic utility function the simple item response model (Rasch, 1961) is mathematically equivalent to the basic spatial model if legislators have quadratic utility functions with additive random error (Ladha, 1991; Londregan, 2000; Clinton, Jackman, and Rivers, 2004). This has the effect of making the estimation quite straightforward as it boils down to a series of linear regressions.

As this is written early in the 21st Century, the influence of psychometrics shows no sign of abating in political science. The level of sophistication of psychometric applications in political science has steadily increased in the past 20 years. The availability of fast computing has opened up whole new areas of research that were impossible to explore as late as the mid 1980s. In addition, political science methodologists have successfully blended methods from statistics and econometrics with psychometrics to produce unique applications. Heretofore "obscure" methods of estimation are being transmitted between neighboring disciplines much more rapidly than ever before by a younger generation of technically trained scholars. This is an exciting time to be active in applied statistical methods in political science. The coming 20 years should see equally important breakthroughs as massively parallel supercomputing becomes widely available and the information revolution increases the speed of transmission of statistical advances to cadres of ever better trained practitioners.

References

Andrich, David. 1978. "Relationships Between the Thurstone and Rasch Approaches to Item Scaling." *Applied Psychological Measurement*, 2:449-460.

Brady, Henry E. 1985a. "Statistical Consistency and Hypothesis Testing for Nonmetric Multidimensional Scaling." *Psychometrika*, 50:503-537.

Brady, Henry E. 1985b. "The Perils of Survey Research: Inter-Personally Incomparable Responses." *Political Methodology*, 11:269–290.

Brady, Henry E. 1989. "Factor and Ideal Point Analysis for Interpersonally Incomparable Data." *Psychometrika*, 54:181–202.

Brady, Henry E. 1990. "Traits Versus Issues: Factor Versus Ideal-Point Analysis of Candidate Thermometer Ratings." *Political Analysis*, 2:97-129.

Cahoon, Lawrence S. 1975. *Locating a Set of Points Using Range Information Only*. Ph.D. Dissertation, Department of Statistics, Carnegie-Mellon University.

Cahoon, Lawrence S., Melvin J. Hinich, and Peter C. Ordeshook. 1976. "A Multidimensional Statistical Procedure for Spatial Analysis." Manuscript. Carnegie-Mellon University.

Cahoon, Lawrence S., Melvin J. Hinich, and Peter C. Ordeshook. 1978. "A Statistical Multidimensional Scaling Method Based on the Spatial Theory of Voting." In *Graphical Representation of Multivariate Data*, edited by P. C. Wang. New York: Academic Press.

Carroll, J. Douglas and Jih-Jie Chang. 1970. "Analysis of Individual Differences in Multidimensional Scaling via an N-Way Generalization of 'Eckart-Young' decomposition." *Psychometrika*, 35:283-320.

Chang, Jih-Jie and J. Douglas Carroll. 1969. "How to Use MDPREF, a Computer Program for Multidimensional Analysis of Preference Data." *Multidimensional Scaling Program Package of Bell Laboratories.* Bell Laboratories, Murray Hill, N.J.

Cheng, Ken. 2000. "Shepard's Universal Law Supported by Honeybees in Spatial Generalization." *Psychological Science*, 5:403-408.

Clinton, Joshua D., Simon D. Jackman, and Douglas Rivers. 2004. "The Statistical Analysis of Roll Call Data: A Unified Approach." *American Political Science Review*, 98:355-370.

Coombs, Clyde. 1950. "Psychological Scaling Without a Unit of Measurement." *Psychological Review*, 57:148-158.

Coombs, Clyde. 1952. "A Theory of Psychological Scaling." *Engineering Research Bulletin Number 34.* Ann Arbor: University of Michigan Press.

Coombs, Clyde. 1958. "On the Use of Inconsistency of Preferences in Psychological Measurement." *Journal of Experimental Psychology*, 55:1-7.

Coombs, Clyde. 1964. *A Theory of Data*. New York: Wiley.

Davis, Otto A. and Melvin J. Hinich. 1966. "A Mathematical Model of Policy Formation in a Democratic Society." In *Mathematical Applications in Political Science, II*, ed. J. L. Bernd. Dallas: SMU Press, pp. 175 – 208.

Davis, Otto A. and Melvin J. Hinich. 1967. "Some Results Related to a Mathematical Model of Policy Formation in a Democratic Society." In *Mathematical Applications in Political Science III*, edited by J. Bernd. Charlottesville, VA: University of Virginia Press, pp. 14-38.

Davis, Otto A., Melvin J. Hinich, and Peter C. Ordeshook. 1970. "An Expository Development of a Mathematical Model of the Electoral Process." *American Political Science Review*, 64: 426-448.

Eckart, Carl H. and Gale Young. 1936. "The Approximation of One Matrix by Another of Lower Rank." *Psychometrika*, 1:211-218.

Enelow, James M. and Melvin Hinich. 1984. *The Spatial Theory of Voting*. New York: Cambridge University Press.

Galton, Francis. 1869. *Hereditary Genius*. London: Macmillan.

Garner, Wendell R. 1974. *The Processing of Information and Structure*. New York: Wiley.

Geman, Donald, and Stuart Geman. 1984. "Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images." *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6:721-741.

Gelfand, Alan E. and Adrian F. M. Smith. 1990. "Sampling-Based Approaches to Calculating Marginal Densities." *Journal of the American Statistical Association*, 85:398-409.

Gelman, Andrew. 1992. "Iterative and Non-iterative Simulation Algorithms." *Computing Science and Statistics*, 24:433-438.

Gelman, Andrew, John B. Carlin, Hal S. Stern, and Donald B. Rubin. 2000. *Bayesian Data Analysis*. New York: Chapman and Hall/CRC.

Gill, Jeff. 2002. *Bayesian Methods: A Social and Behavioral Sciences Approach*. Boca Raton, FL: Chapman & Hall/CRC.

Gluck, Mark A. 1991. "Stimulus Generalization and Representation in Adaptive Network Models of Category Learning." *Psychological Science*, 2:50-55.

Gower, John C. 1966. "Some Distance Properties of Latent Root and Vector Methods Used in Multivariate Analysis." *Biometrika*, 53:325-338.

Guttman, Louis L. 1944. "A Basis for Scaling Qualitative Data." *American Sociological Review*, 9:139-150.

Guttman, Louis L. 1950. "The Basis for Scalogram Analysis. In Stouffer et al. *Measurement and Prediction: The American Soldier Vol. IV*. New York: Wiley.

Hastings, W. Keith. 1970. "Monte Carlo Sampling Methods Using Markov Chains and their Applications." *Biometrika*, 54:97-109.

Heckman, James J. and James M. Snyder. 1997. "Linear Probability Models of the Demand for Attributes With an Empirical Application to Estimating the Preferences of Legislators." *Rand Journal of Economics*, 28:142-189.

Hitchcock, David B. 2003. "A History of the Metropolis-Hastings Algorithm." *The American Statistician*, 57:254-257.

Horst, Paul. 1963. *Matrix Algebra for Social Scientists*. New York: Holt, Rinehart and Winston, Inc.

Hotelling, Harold. 1929. "Stability in Competition." *Economic Journal*, 39:41-57.

Hotelling, Harold. 1931. "The Generalization of Student's Ratio." *Annals of Mathematical Statistics*, 2:360-378.

Hotelling, Harold. 1933. "Analysis of a Complex Statistical Variables with Principal Components." *Journal of Educational Psychology*, 24:498-520.

Jackman, Simon D. 2000a. "Estimation and Inference via Bayesian Simulation: An Introduction to Markov Chain Monte Carlo." *American Journal of Political Science*, 44:375-404.

Jackman, Simon D. 2000b. "Estimation and Inference are 'Missing Data' Problems: Unifying Social Science Statistics via Bayesian Simulation." *Political Analysis*, 8:307-332.

Jackman, Simon D. 2001. "Multidimensional Analysis of Roll Call Data via Bayesian Simulation: Identification, Estimation, Inference and Model Checking." *Political Analysis*, 9:227-241.

Jacoby, William G. 1991. *Data Theory and Dimensional Analysis*. Newbury Park, CA: Sage Publications.

Jensen, Arthur R. 1998. *The g Factor: The Science of Mental Ability*. Westport, CT: Praeger.

Johnson, Richard M. 1963. "On a Theorem Stated by Eckart and Young." *Psychometrika*, 28:259-263.

Keller, Joseph B. 1962. "Factorization of Matrices by Least-Squares." *Biometrika*, 49:239-242.

Kruskal, Joseph B. 1964a. "Multidimensional Scaling by Optimizing a Goodness of Fit to a Nonmetric Hypothesis." *Psychometrika*, 29:1-27.

Kruskal, Joseph B. 1964b. "Nonmetric Multidimensional Scaling: A Numerical Method." *Psychometrika*, 29:115-129.

Kruskal, Joseph B. 1965. "Analysis of Factorial Experiments by Estimating Monotone Transformations of the Data." *Journal of the Royal Statistical Society B*, 27:251-263.

Kruskal, Joseph B. and Myron Wish. 1978. *Multidimensional Scaling*. Beverly Hills, CA: Sage.

Kruskal, Joseph B., Forrest W. Young, and Judith B. Seery. 1973. "How to Use KYST: A Very Flexible Program to Do Multidimensional Scaling and Unfolding." *Multidimensional Scaling Program Package of Bell Laboratories.* Bell Laboratories, Murray Hill, N.J.

Ladha, Krishna K. 1991. "A Spatial Model of Legislative Voting With Perceptual Error." *Public Choice*, 68:151-174.

Lawson, Charles L. and Richard J. Hanson. 1974. *Solving Least Squares Problems*. Englewood Cliffs, N.J.: Prentice-Hall Inc.

Lingoes, James C. 1963. "Multiple Scalogram Analysis: A Set-Theoretic Model for Analyzing Dichotomous Items." *Education and Psychological Measurement*, 23:501-524.

Londregan, John B. 2000. "Estimating Legislators' Preferred Points." *Political Analysis*, 8(1), 35-56.

Lovie, Alexander D. 1995. "Who Discovered Spearman's Rank Correlation?" *British Journal of Mathematical and Statistical Psychology*, 48:255-269.

Lovie, Alexander D. and Patricia Lovie. 1993. "Charles Spearman, Cyril Burt, and the Origins of Factor Analysis." *Journal of the History of the Behavioral Sciences*, 29:308-321.

MacRae, Duncan, Jr. 1958. *Dimensions of Congressional Voting*. Berkeley: University of California Press.

MacRae, Duncan, Jr. 1970. *Issues and Parties in Legislative Voting*. New York: Harper and Row.

McFadden, Daniel. 1976. "Quantal Choice Analysis: A Survey." *Annals of Economic and Social Measurement*, 5:363-390.

Martin, Andrew D. 2003. "Bayesian Inference for Heterogeneous Event Counts." *Sociological Methods and Research*, 32:30-63.

Martin, Andrew D. and Kevin M. Quinn. 2002. "Dynamic Ideal Point Estimation via Markov Chain Monte Carlo for the U.S. Supreme Court, 1953-1999." *Political Analysis*, 10:134-153.

Metropolis, Nicholas C., and Stanislaw Ulam. 1949. "The Monte Carlo Method." *Journal of the American Statistical Association*, 44:335-341.

Munk, Walter H. and Rudolph W. Preisendorfer. 1976. "Carl Henry Eckart." *Biographical Memoirs V.48*, Washington, D.C.: The National Academy of Sciences.

Nosofsky, Robert M. 1984. "Choice, Similarity, and the Context Theory of Classification." *Journal of Experimental Psychology: Learning, Memory and Cognition*, 10:104-114.

Nosofsky, Robert M. 1992. "Similarity Scaling and Cognitive Process Models." *Annual Review of Psychology*, 43:25-53.

Pearson, Karl P. 1901. "On Lines and Planes of Closest Fit to Systems of Points in Space." *The London, Edinburgh and Dublin Philosophical Magazine and Journal*, 6 (Issue 2):559-572.

Poole, Keith T. 1981. "Dimensions of Interest Group Evaluations of the U.S. Senate, 1969-1978." *American Journal of Political Science*. 25:49-67.

Poole, Keith T. 1984. "Least Squares Metric, Unidimensional Unfolding." *Psychometrika*, 49:311-323.

Poole, Keith T. 1990. "Least Squares Metric, Unidimensional Scaling of Multivariate Linear Models." *Psychometrika*, 55: 123-149.

Poole, Keith T. 2005. *Spatial Models of Parliamentary Voting*. New York: Cambridge University Press.

Poole, Keith T. and R. Steven Daniels (1985). "Ideology, Party, and Voting in the U.S. Congress, 1959-80." *American Political Science Review*, 79:373-399.

Poole, Keith T. and Howard Rosenthal. 1984. "U.S. Presidential Elections 1968-1980: A Spatial Analysis." *American Journal of Political Science*, 28:282-312.

Poole, Keith T. and Howard Rosenthal. 1985. "A Spatial Model for Legislative Roll Call Analysis." *American Journal of Political Science*, 29:357-384.

Poole, Keith T. and Howard Rosenthal. 1991. "Patterns of Congressional Voting." *American Journal of Political Science*, 35:228-278.

Poole, Keith T. and Howard Rosenthal. 1997. *Congress: A Political-Economic History of Roll Call Voting*. New York: Oxford University Press.

Quinn, Kevin M. 2004. "Bayesian Factor Analysis for Mixed Ordinal and Continuous Responses." *Political Analysis*, 12:338-353.

Quinn, Kevin M., Andrew D. Martin. 2002. "An Integrated Computational Model of Multiparty Electoral Competition." *Statistical Science*, 17:405-419.

Quinn, Kevin M., Andrew D. Martin, and Andrew B. Whitford. 1999. "Voter Choice in Multi-Party Democracies: A Test of Competing Theories and Models." *American Journal of Political Science*, 43:1231-1247.

Rabinowitz, George. 1973. *Spatial Models of Electoral Choice: An Empirical Analysis*. Doctoral dissertation, University of Michigan.

Rabinowitz, George. 1976. "A Procedure for Ordering Object Pairs Consistent With the Multidimensional Unfolding Model." *Psychometrika*, 45:349-373.

Rasch, Georg. 1961. "On General Laws and the Meaning of Measurement in Psychology." *Proceedings of the IV Berkeley Symposium on Mathematical Statistics and Probability*, 4:321-333.

Ross, John and Norman Cliff. 1964. "A Generalization of the Interpoint Distance Model." *Psychometrika*, 29:167-176.

Rusk, Jerrold and Herbert Weisberg. 1972. "Perceptions of Presidential Candidates." *Midwest Journal of Political Science*, 16:388-410.

Schofield, Norman, Andrew D. Martin, Kevin M. Quinn, and Andrew B. Whitford. 1998. "Multiparty Electoral Competition in the Netherlands and Germany: A Model Based on Multinomial Probit." *Public Choice*. 97: 257-293.

Shepard, Roger N. 1958. "Stimulus and Response Generalization: Deduction of the Generalization Gradient From a Trace Model." *Psychological Review*, 65:242-256.

Shepard, Roger N. 1962a. "The Analysis of Proximities: Multidimensional Scaling With an Unknown Distance Function. I." *Psychometrika*, 27:125-139.

Shepard, Roger N. 1962b. "The Analysis of Proximities: Multidimensional Scaling With an Unknown Distance Function. II." *Psychometrika*, 27: 219-246.

Shepard, Roger N. 1963. "Analysis of Proximities as a Technique for the Study of Information Processing in Man." *Human Factors*, 5:33-48.

Shepard, Roger N. 1987. "Toward a Universal Law of Generalization for Psychological Science." *Science*, 237:1317-1323.

Shepard, Roger N. 1991. "Integrality Versus Separability of Stimulus Dimensions: Evolution of the Distinction and a Proposed Theoretical Basis." In *Perception of Structure*, edited by James R. Pomerantz and Gregory Lockhead. Washington, D.C.: APA.

Shye, Samuel. 1978. *Theory Construction and Data Analysis in the Behavioral Sciences*. San Francisco: Jossey-Bass.

Spearman, Charles E. 1904. "'General Intelligence' Objectively Determined and Measured." *American Journal of Psychology*, 15:201-293.

Spearman, Charles E. 1906. "'Footrule' for Measuring Correlation." *British Journal of Psychology*, 2:89-108.

Spearman, Charles E. 1927. *The Abilities of Man: Their Nature and Measurement*. New York: Macmillan.

Takane, Yoshio. 2004. "Matrices with special reference to applications in psychometrics." *Linear Algebra and Its Applications*, 388C:341-361.

Takane, Yoshio, Forrest W. Young, and Jan de Leeuw. 1977. "Nonmetric Individual Differences Multidimensional Scaling: An Alternating Least-Squares Method With Optimal Scaling Features." *Psychometrika*, 42:7-67.

Thurstone, Lewis L. 1927. "A Law of Comparative Judgment." *Psychological Review*, 34:278-286.

Thurstone, Lewis L. 1931. "Multiple Factor Analysis." *Psychological Review*, 38:406-427.

Thurstone, Lewis L. 1935. *The Vectors of Mind: Multiple Factor Analysis for the Isolation of Primary Traits.* Chicago: University of Chicago Press.

Thurstone, Lewis L. 1947. *Multiple Factor Analysis*. Chicago: University of Chicago Press.

Torgerson, Warren S. 1952. "Multidimensional Scaling: I. Theory and Method. *Psychometrika,* 17:401-419.

Torgerson, Warren S. 1958. *Theory and Methods of Scaling*. New York: Wiley.

Tversky, Amos. 1992. "Clyde Hamilton Coombs." *Biographical Memoirs V.61*, Washington, D.C.: The National Academy of Sciences.

Van Schuur, Wijbrandt H. 1992. "Nonparametric Unidimensional Unfolding for Multicategory Data." In John H. Freeman, editor, *Political Analysis*, volume 4. Ann Arbor, Mich.: University of Michigan Press.

Van Schuur, Wijbrandt H. 2003. "Mokken Scale Analysis: Between the Guttman Scale and Parametric Item Response Theory." *Political Analysis*, 11:139-163.

Weisberg, Herbert F. 1968. *Dimensional Analysis of Legislative Roll Calls*. Doctoral dissertation, University of Michigan.

Weisberg, Herbert F. and Jerrold G. Rusk. 1970. "Dimensions of Candidate Evaluation." *American Political Science Review*, 64:1167-1185.

Young, Forrest W., Jan de Leeuw, and Yoshio Takane. 1976. "Regression With Quantitative and Qualitative Variables: An Alternating Least Squares Method With Optimal Scaling Features." *Psychometrika*, 41:505-529.

Young, Gale and Alston S. Householder. 1938. "Discussion of a Set of Points in Terms of their Mutual Distances." *Psychometrika*, 3:19-22.

---

[1] This rank-order correlation was not what is now known as the Spearman correlation coefficient. He first used the correlation coefficient that would eventually be named for him in a paper published in 1906 (Lovie, 1995).

[2] This formula is detailed in Spearman (1927). For a detailed discussion of Spearman's work on the g factor see Jensen (1998).

[3] The SVD Theorem was stated by Eckart and Young (1936) in their famous paper but they did not provide a proof. The first proof that every *rectangular* matrix of real elements can be decomposed into the product of two orthogonal matrices – **U** and **V** – and a diagonal matrix $\mathbf{\Lambda}$**,** namely, $\mathbf{U\Lambda V'}$ as shown in the statement of the Eckart-Young theorem, was given by Johnson (1963). Horst (1963) refers to the singular value decomposition as the *basic structure* of a matrix and discusses the mechanics of matrix

decomposition in detail in chapters 17 and 18. A more recent treatment can be found in chapters 1 and 2 of Lawson and Hanson (1974).

[4] See van Schuur (1992, 2003) for a discussion of some Guttman-like models. The multidimensional generalization of Guttman scaling is known as Multidimensional Scalogram Analysis (Lingoes, 1963). For a survey see Shye (1978, chapters 9-11).

[5] See Jacoby (1991) for an overview and synthesis of the alternating least squares approach in psychometrics.

[6] The work of Heckman and Snyder (1997) is also based on the spatial model and the random utility model. However, even though it uses principal components analysis, it is more accurately classified as an econometrics method than a psychometrics method.

[7] See Hitchcock (2003) for a short history of MCMC simulation.